

Collision Theory of Complete Multipartite Graphs

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Abstract

A maximal k -colorable graph on n vertices is a complete k -partite graph $K_{a_1, \dots, a_k}$, and its edge count is $\binom{n}{2} - \frac{1}{2} \sum a_i^2$. Two partitions of n therefore *collide* — produce the same graph size — exactly when they share the value of $\sum a_i^2$. Continuing [4], which studied the set $\Sigma_\chi(n, k)$ of realized edge counts, we study the *multiplicity* $m(e)$ of an edge count and develop a structural theory of collisions. The discrepancy identity $k \sum a_i^2 - n^2 = \sum_{i < j} (a_i - a_j)^2$ places the subject inside the arithmetic of the root lattice A_{k-1} , and the theory changes character with k exactly as the theory of quadratic forms in $k - 1$ variables does. Our main results are: (i) a hidden involution $a_i \mapsto 2n/k - a_i$, defined whenever $k \mid 2n$, which forces collisions and yields $\limsup |\Sigma_\chi(n, 3)|/p_3(n) \leq 2/3$ along $3 \mid n$; (ii) for $k = 3$, an exact multiplicity formula $m = E(N) + [N = \square]$ (halved when $3 \nmid n$), where $N = (3s - n^2)/2$ and E is the Eisenstein ideal-counting function, verified exhaustively on 386 fibers; (iii) the maximal multiplicity asymptotic $\log M(n, 3) = (2 \log 2 + o(1)) \log n / \log \log n$; (iv) the density theorem $|\Sigma_\chi(n, 3)| \asymp n^2 / \sqrt{\log n}$, which shows that the collision-density ratios observed numerically in [4] are transient and tend to 0, resolving [4, Question 10.2]; (v) a phase transition $M(n, k) = n^{k-3+o(1)}$ for $k \geq 4$; (vi) for $k = 4$, unconditional spectrum gaps governed by the three-squares theorem: no discrepancy of the form $4^c(8b+7)$ is ever realized; (vii) a complete answer to [4, Question 10.3]: for $k \geq 3$ the two largest spectrum values always differ by exactly 1; and (viii) a *depth filtration* identifying K_t -subgraph counts with elementary symmetric polynomials, linking deep collisions to the Prouhet–Tarry–Escott problem and producing non-isomorphic complete 4-partite graphs on 18 vertices with identical vertex, edge, and triangle counts. We close with a study of collision dynamics and several open problems.

1 Introduction

Let $\chi(G)$ denote the chromatic number. A graph G on n vertices is *maximal k -colorable* if $\chi(G) = k$ and adding any edge raises the chromatic number; classically these are exactly the complete k -partite graphs $K_{a_1, \dots, a_k}$ (see [4, Theorem 3.2]). Since

$$e(K_{a_1, \dots, a_k}) = \binom{n}{2} - \frac{1}{2} \sum_{i=1}^k a_i^2, \quad (1)$$

the edge count of such a graph is an affine function of the power sum $s(a) = \sum a_i^2$ of the underlying partition. The paper [4] introduced the *edge spectrum* $\Sigma_\chi(n, k) = \{e(G) : G \text{ maximal } k\text{-colorable on } n \text{ vertices}\}$, determined the extremal values $\text{sat}_\chi(n, k)$ and $\text{ab}_\chi(n, k)$, described $\Sigma_\chi(n, 2)$ completely, and discovered that for $k = 3$ distinct partitions may produce equal edge counts — the first *collision* occurring at $n = 9$, where $(1, 4, 4)$ and $(2, 2, 5)$ both give 24 edges

([4, Proposition 7.1]). Numerical experiments in [4, Remark 7.4] suggested that the collision density $|\Sigma_\chi(n, 3)|/p_3(n)$ stabilizes near 0.76 for $3 \nmid n$ and near 0.55 for $3 \mid n$, and [4] posed several questions about this phenomenon.

The present paper makes the collision phenomenon itself the object of study. For a partition $a = (a_1, \dots, a_k)$ of n into k positive parts, define the *multiplicity*

$$m_{n,k}(e) = \#\{a \vdash n \text{ into } k \text{ parts} : e(K_a) = e\}, \quad M(n, k) = \max_e m_{n,k}(e),$$

so that $|\Sigma_\chi(n, k)| = \#\{e : m_{n,k}(e) \geq 1\}$ and $\sum_e m_{n,k}(e) = p_k(n)$, the number of partitions of n into k parts. By (1), $m_{n,k}$ is equally the multiplicity function of the statistic $s(a) = \sum a_i^2$ on partitions, and we will pass freely between e and s .

Our starting point is an elementary identity (Lemma 2.1) which converts the statistic s into a quadratic form on the root lattice A_{k-1} : collisions are multiple representations of an integer by that form, subject to a congruence condition recording $n \bmod k$ and a positivity condition confining the representation to a simplex. This single observation imports into extremal graph theory the classical trichotomy of quadratic forms in $d = k - 1$ variables — binary forms ($k = 3$, governed by ideal theory in an imaginary quadratic field), ternary forms ($k = 4$, governed by the three-squares theorem and class numbers), and forms in four or more variables ($k \geq 5$, governed by the circle method) — and the collision theory faithfully reproduces this trichotomy.

1.1 Main results

The reflection involution (Section 3). Whenever $k \mid 2n$, the map $\iota(a_1, \dots, a_k) = (2n/k - a_1, \dots, 2n/k - a_k)$ preserves both $\sum a_i$ and $\sum a_i^2$ exactly (Theorem 3.1). For $k = 2$ the involution degenerates to a transposition, which explains the injectivity proved in [4, Theorem 6.4(b)]; for $k = 3$ it exists precisely when $3 \mid n$, forces the pairing $(1, 4, 4) \leftrightarrow (2, 2, 5)$ at $n = 9$, and yields the unconditional bound $\limsup_{3 \mid n} |\Sigma_\chi(n, 3)|/p_3(n) \leq 2/3$ (Theorem 3.3), consistent with the value ≈ 0.55 observed in [4].

Exact multiplicity for $k = 3$ (Section 4). Setting $N = (3s - n^2)/2$, collisions for $k = 3$ are multiple representations of N by the Eisenstein norm form $x^2 + xy + y^2$. With $E(N) = \sum_{d \mid N} \chi_{-3}(d)$ the ideal-counting function of $\mathbb{Z}[\omega]$ and $w(N) = E(N) + [N \text{ a square}]$, we prove (Theorem 4.5) that always $m \leq w(N)$ (resp. $w(N)/2$ when $3 \nmid n$), with *equality* for all spectrum values away from the boundary. The formula postdicts every collision in the tables of [4] and predicts, correctly, that the first triple collision occurs at $n = 27$ and the first quadruple at $n = 36$. Two asymptotic consequences follow:

$$\log M(n, 3) = (2 \log 2 + o(1)) \frac{\log n}{\log \log n} \quad (\text{Theorem 4.8}),$$

so multiplicities for $k = 3$ are unbounded but sub-polynomial, and

$$|\Sigma_\chi(n, 3)| \asymp \frac{n^2}{\sqrt{\log n}} \quad (\text{Theorem 4.10}),$$

by a Landau–Bernays count of the integers represented by $x^2 + xy + y^2$. In particular $|\Sigma_\chi(n, 3)|/p_3(n) \rightarrow 0$: the plateaus at 0.76 and 0.55 reported in [4, Remark 7.4] are a pre-asymptotic mirage produced by the glacial decay rate $1/\sqrt{\log n}$, and we quantify why no computation at the scale $n \leq 300$ could have detected the decay (Observation 4.12). This resolves [4, Question 10.2] (the

limits are 0; the residue dependence is real at every finite scale and is explained by ramification of the prime 3) and settles the $k = 3$ case of [4, Question 8.2].

The spectrum for $k \geq 4$ (Section 5). We answer [4, Question 8.2] in full: $|\Sigma_\chi(n, k)| = \Theta(n^{k-1})$ fails for every $k \geq 3$, and in fact $|\Sigma_\chi(n, k)| = \Theta_k(n^2)$ for all $k \geq 5$ and for $k = 4$ when $4 \mid n$ (Theorem 5.4). For $k = 4$ we prove an unconditional gap theorem via Gauss–Legendre: no edge count whose discrepancy $D = 4 \sum a_i^2 - n^2$ has the form $4^c(8b+7)$ is ever realized (Theorem 5.7); computations indicate these are the *only* bulk exclusions. For $k \geq 8$ with $k \mid n$ we prove by contrast that the top of the spectrum is a complete integer interval of length $\Theta(n^2/k^2)$ (Theorem 5.10), supporting [4, Conjecture 10.1]. We also resolve [4, Question 10.3] completely: for every $k \geq 3$ and $n \geq k+2$ the two largest elements of $\Sigma_\chi(n, k)$ differ by exactly 1, while for $k = 2$ they differ by 1 precisely when n is even (Theorem 5.1). Finally, the maximal multiplicity undergoes a phase transition (Theorem 5.14):

$$M(n, 2) = 1, \quad M(n, 3) = e^{\Theta(\log n / \log \log n)}, \quad M(n, k) = n^{k-3+o(1)} \quad (k \geq 4),$$

where the lower bounds are elementary (pigeonhole against the quadratic width of the spectrum) and the upper bounds come from lattice point counts on spheres.

The depth filtration (Section 6). The number of K_t -subgraphs of $K_{a_1, \dots, a_k}$ is the elementary symmetric polynomial $e_t(a_1, \dots, a_k)$. Consequently two complete k -partite graphs agree on all clique counts $K_1, \dots, K_r$ iff their partitions share the power sums $p_1, \dots, p_r$, and collisions acquire a *depth*: collisions of depth r are exactly size- k solutions of the Prouhet–Tarry–Escott problem of degree r . The smallest deep example is striking: $K_{1,4,5,8}$ and $K_{2,2,7,7}$ are non-isomorphic graphs on 18 vertices with 109 edges and 252 triangles each, first distinguished by their K_4 -counts (Example 6.4).

Collision dynamics (Section 7). Reflection moves (local versions of ι) generate many collisions but provably not all: for $k = 3$ the unreachable collisions are precisely the *arithmetic* ones, arising from inequivalent Eisenstein ideals of equal norm, and we show that adjoining a conjugation move renders every interior $k = 3$ fiber connected (Proposition 7.4). For $k \geq 4$ the analogous completeness question is open and connects to the genus theory of ternary forms.

Throughout, every claim verified by machine rather than proof is labelled an Observation, and Section 8 documents the exhaustive computations, including the searches for counterexamples to each theorem.

1.2 Relation to prior work

The multiplicity of edge counts of complete multipartite graphs appears not to have been studied before, but the ingredients we use are classical: representations by binary quadratic forms and the Landau–Bernays density theorem [11, 2, 12]; the arithmetic of the Eisenstein integers [10]; the three-squares theorem and lattice identifications $A_3 \cong D_3$ [5]; equidistribution of lattice points on spheres [7, 8]; the maximal order of the divisor function [9]; and the Prouhet–Tarry–Escott problem [3]. The saturation-spectrum literature surveyed in [4] concerns which sizes occur; our subject is how often.

2 The discrepancy and the lattice model

All partitions in this paper have exactly k positive parts, written $a = (a_1, \dots, a_k)$ with $a_1 \leq \dots \leq a_k$ and $\sum a_i = n$; $p_k(n)$ denotes their number, and $p_k(n) \sim n^{k-1}/(k!(k-1)!)$ for fixed k

[1]. We write $s(a) = \sum a_i^2$, so that $e(K_a) = \binom{n}{2} - \frac{1}{2}s(a)$, and define the *discrepancy*

$$D(a) = k s(a) - n^2.$$

Lemma 2.1 (Discrepancy identity). *For any real numbers $a_1, \dots, a_k$ with $\sum a_i = n$,*

$$k \sum_i a_i^2 - n^2 = \sum_{i < j} (a_i - a_j)^2.$$

Proof. $\sum_{i < j} (a_i - a_j)^2 = (k-1) \sum a_i^2 - 2 \sum_{i < j} a_i a_j = (k-1) \sum a_i^2 - (n^2 - \sum a_i^2) = k \sum a_i^2 - n^2$. $\square$

Thus $D(a) \geq 0$, with equality iff all parts are equal, and two partitions of n into k parts collide iff they have equal discrepancy. Two elementary parity facts will be used repeatedly.

Lemma 2.2. *For every partition a of n into k parts, $s(a) \equiv n \pmod{2}$. In particular all elements of the fiber structure live in a single parity class of s , and consecutive realizable values of s differ by at least 2.*

Proof. $a_i^2 \equiv a_i \pmod{2}$, so $s(a) \equiv \sum a_i = n$. $\square$

Theorem 2.3 (Lattice model). *Let $\Lambda_{k-1} = \{v \in \mathbb{Z}^k : \sum v_i = 0\}$, the root lattice A_{k-1} , with the quadratic form $|v|^2 = \sum v_i^2$, and let $W \cong S_k$ act by permuting coordinates. The map $a \mapsto w = ka - n\mathbf{1}$ is a bijection between partitions of n into k positive parts and W -orbits of vectors $w \in \Lambda_{k-1}$ satisfying the congruence $w \equiv -n\mathbf{1} \pmod{k}$ and the positivity condition $\min_i w_i \geq k - n$. Under this bijection $|w|^2 = k D(a)$. Consequently $m_{n,k}$ is a representation number of the A_{k-1} quadratic form restricted to a congruence coset and a dilated simplex, and away from the simplex boundary the generating function of the multiplicities is a coset theta series of A_{k-1} , a modular form of weight $(k-1)/2$.*

Proof. If $\sum a_i = n$ then $\sum w_i = kn - kn = 0$ and $w \equiv -n\mathbf{1} \pmod{k}$; positivity $a_i \geq 1$ is $w_i \geq k - n$. Conversely any such w gives $a = (w + n\mathbf{1})/k$, integral by the congruence, summing to n , positive by the inequality. Unordered partitions correspond to W -orbits. Finally $|w|^2 = \sum (ka_i - n)^2 = k^2 s(a) - 2kn \cdot n + kn^2 = k(ks(a) - n^2) = kD(a)$. The modularity statement is the classical modularity of theta series of lattice cosets [5, Ch. 4]; we use it only as orientation and never rely on it in proofs. $\square$

Remark 2.4. The weight ladder $(k-1)/2 = 1, \frac{3}{2}, 2, \dots$ already predicts the shape of the theory: weight 1 ($k=3$) means multiplicities are divisor-type sums over an imaginary quadratic field; weight $\frac{3}{2}$ ($k=4$) means class numbers, the three-squares theorem, and (for equidistribution) Duke's theorem [7]; weight ≥ 2 ($k \geq 5$) means Eisenstein-dominated growth. The rest of the paper substantiates each rung, mostly by elementary or classical means.

Remark 2.5 (Generating function). Writing $G(q, z) = \sum_{a \geq 1} q^a z^{a^2}$, a partial theta function, the composition version of the multiplicity generating function is $\sum_{n,s} c_{n,k}(s) q^n z^s = G(q, z)^k$, where $c_{n,k}(s)$ counts *ordered* k -tuples; the unordered version is obtained by substituting $G(q^j, z^j)$ into the cycle index of S_k . We record this for future use and do not pursue it here.

3 The reflection involution

The Weyl group $W \cong S_k$ has index 2 in $\text{Aut}(A_{k-1})$ for $k \geq 3$; the missing symmetry is $w \mapsto -w$. Pulled back through Theorem 2.3 it becomes a concrete and, to our knowledge, unremarked involution on partitions.

Theorem 3.1 (Reflection involution). *Suppose $k \mid 2n$ and set $t = 2n/k$. Then $\iota(a_1, \dots, a_k) = (t - a_1, \dots, t - a_k)$ satisfies $\sum(t - a_i) = n$ and $\sum(t - a_i)^2 = \sum a_i^2$. Hence if all parts of a are at most $t - 1$ and the multiset a is not invariant under $x \mapsto t - x$, then $\{a, \iota(a)\}$ is a collision pair.*

Proof. $\sum(t - a_i) = kt - n = 2n - n = n$, and $\sum(t - a_i)^2 = kt^2 - 2t \sum a_i + \sum a_i^2 = \frac{4n^2}{k} - \frac{4n^2}{k} + \sum a_i^2$. The last sentence is immediate since $\iota(a)$ is then again a partition of n into k positive parts, distinct from a , with the same s . $\square$

Remark 3.2. For $k = 2$, $t = n$ and $\iota(a, n - a) = (n - a, a)$: the involution degenerates to the transposition, producing no collisions. This is precisely why $\Sigma_\chi(n, 2)$ has all multiplicities equal to 1 ([4, Theorem 6.4(b)]); the collision phenomenon begins at $k = 3$ because that is where -1 first fails to lie in the Weyl group. For $k = 3$ the involution exists iff $3 \mid n$, and at $n = 9$ it sends $(1, 4, 4) \mapsto (5, 2, 2)$: the first collision found in [4, Proposition 7.1] is a reflection collision.

Theorem 3.3. $\limsup_{\substack{n \rightarrow \infty \\ 3 \mid n}} \frac{|\Sigma_\chi(n, 3)|}{p_3(n)} \leq \frac{2}{3}$.

Proof. Let $3 \mid n$ and $t = 2n/3$. Call a partition *movable* if all parts are $\leq t - 1$ and it is not ι -invariant. Movable partitions come in disjoint ι -pairs with equal s , so the number of fibers satisfies $|\Sigma_\chi(n, 3)| \leq p_3(n) - \frac{1}{2} \# \{\text{movable}\}$.

A partition (a, b, c) , $a \leq b \leq c$, fails the size condition iff $c \geq t$, i.e. $a + b \leq n/3$; the number of such pairs (a, b) with $1 \leq a \leq b$ is $\frac{1}{2} \cdot \frac{1}{2} (n/3)^2 + O(n) = n^2/36 + O(n)$. A partition is ι -invariant iff its multiset is symmetric about $n/3$; such a multiset either equals $(n/3, n/3, n/3)$ or consists of $n/3$ together with a pair $\{x, t - x\}$, giving $O(n)$ partitions in all. Since $p_3(n) = n^2/12 + O(n)$ [1],

$$|\Sigma_\chi(n, 3)| \leq p_3(n) - \frac{1}{2} \left(p_3(n) - \frac{n^2}{36} - O(n) \right) = \frac{n^2}{24} + \frac{n^2}{72} + O(n) = \frac{n^2}{18} + O(n),$$

and $(n^2/18)/(n^2/12) = 2/3$. $\square$

The bound is not tight asymptotically (Theorem 4.10 gives $o(1)$), but it is effective at every finite n : at $n = 9$ and $n = 12$ the inequality $|\Sigma_\chi| \leq p_3 - \frac{1}{2} \# \{\text{movable}\}$ holds with equality ($6 \leq 6$ and $10 \leq 10$), and it already explains, at the level of a two-line argument, why the spectrum is systematically smaller when $3 \mid n$, the phenomenon observed empirically in [4, Remark 7.4].

Definition 3.4 (Local reflections). For $k \geq 4$, $S \subseteq \{1, \dots, k\}$ with $|S| = j \geq 3$, and $m = \sum_{i \in S} a_i$ with $j \mid 2m$, the *local reflection* ι_S replaces a_i by $2m/j - a_i$ for $i \in S$ and fixes the other parts, provided all new parts are positive. By the computation of Theorem 3.1 applied to the sub-multiset, ι_S preserves n and s . (For $j \leq 2$ the map is a permutation.)

Example 3.5. At $n = 12$, $k = 4$, the fiber at $s = 42$ is $\{(1, 3, 4, 4), (2, 2, 3, 5)\}$; the local reflection on the sub-multiset $(1, 4, 4)$, of sum 9, sends it to $(5, 2, 2)$ and connects the pair. Local reflections make collisions generic for $k \geq 4$; Section 7 determines how much of the collision relation they generate (not all of it).

4 Exact collision theory for $k = 3$: Eisenstein arithmetic

Throughout this section $k = 3$. For an *ordered* triple (a, b, c) of positive integers with $a + b + c = n$ put

$$x = a - b, \quad y = b - c.$$

Lemma 4.1 (Parametrization). *With $Q(x, y) = x^2 + xy + y^2$ we have $D(a, b, c) = 2Q(x, y)$; equivalently, if $s = a^2 + b^2 + c^2$ then*

$$N := \frac{3s - n^2}{2} = Q(x, y) \in \mathbb{Z}_{\geq 0}.$$

Conversely, a pair $(x, y) \in \mathbb{Z}^2$ arises from an ordered triple summing to n if and only if $x - y \equiv n \pmod{3}$, in which case the triple is uniquely $b = (n - x + y)/3$, $a = b + x$, $c = b - y$, and it has positive parts iff $b \geq 1 + \max(0, -x, y)$. Moreover

$$Q(x, y) \equiv (x - y)^2 \pmod{3}, \tag{2}$$

so every realized invariant satisfies $N \equiv n^2 \pmod{3}$; in particular $N \equiv 0 \pmod{3}$ when $3 \mid n$ and $N \equiv 1 \pmod{3}$ otherwise.

Proof. By Lemma 2.1, $D = (a - b)^2 + (b - c)^2 + (a - c)^2 = x^2 + y^2 + (x + y)^2 = 2Q(x, y)$, and $D = 3s - n^2$. The linear system $a - b = x$, $b - c = y$, $a + b + c = n$ has the stated unique rational solution, integral iff $3 \mid n - (x - y)$. Finally $Q(x, y) - (x - y)^2 = 3xy$, giving (2), and $x - y \equiv n \pmod{3}$ gives $N \equiv n^2$. $\square$

We now recall the arithmetic of the form Q . Let $\omega = e^{2\pi i/3}$ and $\mathbb{Z}[\omega]$ the Eisenstein integers, a principal ideal domain with unit group of order 6 and norm form equivalent to Q ; the prime 3 ramifies as $3 = -\omega^2 \lambda^2$ with $\lambda = 1 - \omega$. Let χ_{-3} be the nontrivial character mod 3 and

$$E(N) = \sum_{d \mid N} \chi_{-3}(d),$$

the number of ideals of $\mathbb{Z}[\omega]$ of norm N ; it is multiplicative with $E(p^\alpha) = \alpha + 1$ for $p \equiv 1 \pmod{3}$, $E(p^\alpha) = \frac{1}{2}(1 + (-1)^\alpha)$ for $p \equiv 2 \pmod{3}$, and $E(3^\alpha) = 1$. The number of representations of $N \geq 1$ by Q is

$$r_Q(N) = 6 E(N); \tag{3}$$

see [10, Ch. 9] (the count of elements of norm N equals the number of ideals times the 6 units).

Next, the relevant symmetry groups. Let $G = \text{Aut}(\mathbb{Z}^2, Q)$, of order 12 (the point group of the hexagonal lattice), and let $W \leq G$ be the subgroup induced by permutations of the ordered triple (a, b, c) through the parametrization of Lemma 4.1.

Lemma 4.2. *W has order 6 and consists of the identity, the rotations $\rho^{\pm 2}$ where $\rho(x, y) = (-y, x + y)$ is the order-6 rotation, and the three reflections*

$$\sigma_1(x, y) = (-x, x + y), \quad \sigma_2(x, y) = (x + y, -y), \quad \sigma_3(x, y) = (-y, -x),$$

whose mirror lines are $\{x = 0\}$, $\{y = 0\}$, $\{x + y = 0\}$ respectively. Moreover $G = W \sqcup (-1)W$, so $-1 \notin W$; the residue $x - y \pmod{3}$ is constant on W -orbits; and $(x - y)(-v) = -(x - y)(v)$.

Proof. The transpositions $a \leftrightarrow b$, $b \leftrightarrow c$, $a \leftrightarrow c$ act on $(x, y) = (a - b, b - c)$ as $\sigma_1, \sigma_2, \sigma_3$ respectively, and the 3-cycles as $\rho^{\pm 2}$; each visibly preserves Q , and the six maps are distinct, so $|W| = 6$. The point group of the hexagonal lattice has order 12 and contains -1 ; since $-\sigma_3(x, y) = (y, x)$ interchanges the mirror families, $-1 \notin W$ and $G = W \sqcup (-1)W$. If $g \in W$ is induced by a permutation of a triple with integer entries then, for any (x, y) with $x - y \equiv n \pmod{3}$, the image also parametrizes an integer triple summing to n , whence $x - y \equiv n \pmod{3}$ again; as this holds for a full congruence class of (x, y) and the action is linear, $x - y \pmod{3}$ is W -invariant. The last assertion is trivial. $\square$

Definition 4.3. For $N \geq 1$ let $w(N)$ denote the number of W -orbits of representations of N by Q .

Lemma 4.4. $w(N) = E(N) + [N \text{ is a perfect square}]$.

Proof. A representation has nontrivial W -stabilizer iff it lies on a mirror of W , i.e. $x = 0$, $y = 0$, or $x + y = 0$; in each case $N = y^2, x^2, x^2$ respectively, so mirror representations exist iff $N = t^2$ and are exactly the six points $(\pm t, 0), (0, \pm t), (t, -t), (-t, t)$. These six split into two W -orbits of size 3, namely $O_+ = \{(t, 0), (-t, t), (0, -t)\}$ and $O_- = -O_+$ (a direct check with Lemma 4.2). All other representations have trivial stabilizer and lie in W -orbits of size 6. By (3) the total number of representations is $6E(N)$; if N is not a square this gives $E(N)$ orbits, and if $N = t^2$ it gives $2 + (6E(N) - 6)/6 = E(N) + 1$ orbits. $\square$

Theorem 4.5 (Multiplicity formula). *Fix $n \geq 4$ and a spectrum value $s \in \{s(a)\}$ with $N = (3s - n^2)/2 \geq 1$, and write $m = m_{n,3}$ for the multiplicity of the corresponding edge count. Then:*

- (a) (Unconditional upper bound.) $m \leq w(N)$ if $3 \mid n$, and $m \leq w(N)/2$ if $3 \nmid n$ (in which case $w(N)$ is even).
- (b) (Interior equality.) If moreover $N \leq 3(n - 3)^2/100$, then equality holds:

$$m = \begin{cases} w(N) = E(N) + [N = \square], & 3 \mid n, \\ \frac{1}{2}w(N) = \frac{1}{2}(E(N) + [N = \square]), & 3 \nmid n. \end{cases}$$

Proof. By Lemma 4.1, unordered partitions in the fiber of s correspond bijectively to W -orbits of representations (x, y) of N that satisfy the congruence $x - y \equiv n \pmod{3}$ (admissible orbits; admissibility is well defined by Lemma 4.2) and whose associated triple has positive parts (positivity is a property of the orbit, since its members are the reorderings of one triple). This proves at once that m is at most the number of admissible orbits.

Count admissible orbits. If $3 \mid n$ then $N \equiv 0 \pmod{3}$ by Lemma 4.1, so by (2) every representation has $x - y \equiv 0 \equiv n \pmod{3}$: all $w(N)$ orbits are admissible. If $3 \nmid n$ then $N \equiv 1 \pmod{3}$ and every representation has $x - y \not\equiv 0 \pmod{3}$. The map $v \mapsto -v$ permutes W -orbits (as $-1 \notin W$ normalizes W), negates $x - y$, and fixes no orbit here: a fixed orbit would contain v and $-v$, forcing $x - y \equiv -(x - y)$, i.e. $3 \mid x - y$, excluded. Hence the orbits split evenly between the residues $x - y \equiv 1$ and $\equiv 2 \pmod{3}$, so exactly $w(N)/2$ are admissible (and $w(N)$ is even). This proves (a).

For (b) it remains to show that under the stated bound every admissible orbit satisfies positivity. If $Q(x, y) = N$ then $N = (x + \frac{y}{2})^2 + \frac{3}{4}y^2 \geq \frac{3}{4}y^2$, so $|y| \leq 2\sqrt{N/3}$, and symmetrically $|x| \leq 2\sqrt{N/3}$. Then $b = (n - x + y)/3 \geq (n - 4\sqrt{N/3})/3$ and $\min(a, b, c) \geq b - \max(0, -x, y) \geq \frac{1}{3}(n - 4\sqrt{N/3}) - 2\sqrt{N/3} = \frac{1}{3}(n - 10\sqrt{N/3})$, which is ≥ 1 as soon as $10\sqrt{N/3} \leq n - 3$, i.e. $N \leq 3(n - 3)^2/100$. $\square$

Remark 4.6. The threshold $3(n-3)^2/100$ is a convenient sufficient bound, not sharp: positivity typically holds far beyond it. The formula was verified by exhaustive computation on all 386 fibers with $N \leq (n-3)^2/12$ for fifteen values of n up to 91, with no exceptions (Section 8).

Example 4.7. (i) $n = 9$: the collision of [4, Proposition 7.1] has $s = 33$, $N = 9$, $w(9) = E(9) + 1 = 1 + 1 = 2$, and $3 \mid 9$, so $m = 2$: the pair $(1, 4, 4), (2, 2, 5)$, which are ι -partners (Remark 3.2). (ii) $n = 22$: the fiber $\{(5, 5, 12), (3, 8, 11)\}$ has $N = 49$, $w(49) = E(49) + 1 = 3 + 1 = 4$, and $3 \nmid 22$, so $m = 4/2 = 2$. Here the two partitions come from *inequivalent* divisors of $49 = \pi^2 \bar{\pi}^2$ (with $7 = \pi \bar{\pi}$), a genuinely arithmetic collision; see Section 7. (iii) The first multiplicity 3: one needs $w \geq 3$ with $3 \mid n$, and the smallest suitable invariant is $N = 3 \cdot 7^2 = 147$ with $E = 3$; this is realized at $n = 27$, $s = 341$, by $(1, 12, 14), (2, 9, 16), (4, 6, 17)$. The first multiplicity 4 arises from $N = 3 \cdot 7 \cdot 13 = 273$, $E = 4$, at $n = 36$. Both were found by search exactly where the formula predicts.

4.1 Maximal multiplicity

Theorem 4.8. $\log M(n, 3) = (2 \log 2 + o(1)) \frac{\log n}{\log \log n}$.

Proof. Upper bound. Every partition of n into 3 parts has $D \leq (n-3)^2 + (n-3)^2 = 2(n-3)^2$ (attained by $(1, 1, n-2)$), so every realized invariant satisfies $N \leq (n-3)^2$. By Theorem 4.5(a), $M(n, 3) \leq \max_{N \leq (n-3)^2} (E(N) + 1) \leq \max_{N \leq n^2} d(N) + 1$, where d is the divisor function, since $E(N) \leq d(N)$. By Wigert's theorem on the maximal order of d [9, Thm. 317], $\log d(N) \leq (\log 2 + o(1)) \log N / \log \log N$, and with $N \leq n^2$ this gives $\log M(n, 3) \leq (2 \log 2 + o(1)) \log n / \log \log n$.

Lower bound. Let $p_1 < p_2 < \dots$ be the primes $\equiv 1 \pmod{3}$ and $N_r = p_1 \cdots p_r$, so $E(N_r) = 2^r$. By the prime number theorem in arithmetic progressions [6], $\sum_{p \leq x, p \equiv 1(3)} \log p \sim x/2$ and $\#\{p \leq x : p \equiv 1(3)\} \sim x/(2 \log x)$, whence $\log N_r = (1 + o(1)) r \log r$. Given n , choose the largest r with $3N_r \leq 3(n-3)^2/100$; then $r = (2 + o(1)) \log n / \log \log n$. If $3 \mid n$ take $N = 3N_r$: since $E(3M) = E(M)$ for $3 \nmid M$, Theorem 4.5(b) gives a fiber of multiplicity $\geq E(3N_r) = 2^r$. If $3 \nmid n$ take $N = N_r$ (note $N_r \equiv 1(3)$ as required): the multiplicity is $\geq \frac{1}{2} E(N_r) = 2^{r-1}$. In both cases $\log M(n, 3) \geq (r-1) \log 2 = (2 \log 2 + o(1)) \log n / \log \log n$. $\square$

Thus for $k = 3$ multiplicities are unbounded but grow more slowly than any power of n ; the measured values ($M = 3, 4, 6, 6, 8, 8$ at $n = 30, 60, \dots, 180$) crawl accordingly. This will contrast sharply with $k \geq 4$ (Theorem 5.14).

4.2 The density theorem and the resolution of [4, Question 10.2]

Call $N \geq 1$ *Loeschian* if it is represented by Q , and let $B(X) = \#\{N \leq X : N \text{ Loeschian}\}$. The Landau–Bernays theorem [11, 2] for the discriminant -3 (see [12] for the constant) gives

$$B(X) \sim K \frac{X}{\sqrt{\log X}}, \quad K = 0.63890 \dots \quad (4)$$

Lemma 4.9 (Ramification). *For $M \geq 1$: $3M$ is Loeschian iff M is Loeschian.*

Proof. If $M = z\bar{z}$ then $3M = (\lambda z)(\bar{\lambda} \bar{z})$. Conversely if $3 \mid z\bar{z}$ then the prime λ divides z or $\bar{z}$; since $\bar{\lambda}$ is an associate of λ , in either case $\lambda \mid z$, so $z = \lambda z'$ and $z\bar{z} = 3 z' \bar{z}'$. $\square$

Theorem 4.10. *There are constants $c_2 > c_1 > 0$ with*

$$c_1 \frac{n^2}{\sqrt{\log n}} \leq |\Sigma_\chi(n, 3)| \leq c_2 \frac{n^2}{\sqrt{\log n}} \quad (n \geq n_0).$$

Proof. Upper bound. Distinct spectrum values have distinct invariants $N \leq (n-3)^2$, all Loeschian (plus possibly $N = 0$), so $|\Sigma_\chi(n, 3)| \leq B(n^2) + 1 \ll n^2/\sqrt{\log n}$ by (4).

Lower bound. Put $Y = 3(n-3)^2/100$. By Theorem 4.5(b), every Loeschian $N \leq Y$ with $N \equiv n^2 \pmod{3}$ is realized (the number of admissible orbits, $w(N)$ or $w(N)/2 \geq 1$, is positive), and distinct N give distinct spectrum values. If $3 \mid n$ the count of realized invariants is at least $\#\{3M \leq Y : M \text{ Loeschian}\} = B(Y/3)$ by Lemma 4.9. If $3 \nmid n$, note first that a Loeschian number is never $\equiv 2 \pmod{3}$ by (2); hence $\#\{N \leq Y \text{ Loeschian}, N \equiv 1 \pmod{3}\} = B(Y) - B(Y/3)$, again by Lemma 4.9. By (4) both counts are $\gg n^2/\sqrt{\log n}$. $\square$

Corollary 4.11. $\frac{|\Sigma_\chi(n, 3)|}{p_3(n)} \rightarrow 0$ as $n \rightarrow \infty$, through every residue class of n . Consequently the limits asked for in [4, Question 10.2] exist and equal 0, and [4, Question 8.2] has a negative answer for $k = 3$: $|\Sigma_\chi(n, 3)| = \Theta(n^2/\sqrt{\log n})$, not $\Theta(n^2)$. Moreover the average multiplicity over realized values diverges: $p_3(n)/|\Sigma_\chi(n, 3)| \asymp \sqrt{\log n}$ — collisions are asymptotically the rule, not the exception.

Observation 4.12 (The pre-asymptotic mirage). The decay in Corollary 4.11 is numerically invisible at the scale of [4, Remark 7.4]. Exhaustive computation gives:

n	$p_3(n)$	$ \Sigma_\chi(n, 3) $	ratio
300	7500	3982	0.5309
301	7550	5743	0.7607
900	67500	32537	0.4820
901	67650	47757	0.7059
2400	480000	215343	0.4486
2401	480400	319802	0.6657

The ratios do decay, but on the schedule $c/\sqrt{\log n}$. Quantitatively, the upper bound $|\Sigma_\chi(n, 3)| \leq B(n^2)$ with the constant (4) only forces the $3 \nmid n$ ratio below the observed 0.76 once $\sqrt{2 \log n} \gtrsim 6.7$, i.e. $n \gtrsim e^{22}$; no table computed at $n \leq 300$, as in [4], could have detected the decay. The residue-3 dependence, by contrast, is visible at every finite scale and is structural: for $3 \mid n$ the realized invariants are $\{3M : M \text{ Loeschian}\}$ (ramification, Lemma 4.9), a Landau-sparsier set than the class $N \equiv 1 \pmod{3}$ available when $3 \nmid n$.

Conjecture 4.13. $\lim_{n \rightarrow \infty} \frac{|\Sigma_\chi(3n', 3)|/p_3(3n')}{|\Sigma_\chi(n'', 3)|/p_3(n'')} = \frac{1}{2}$ (limits over $3 \mid n$ and $3 \nmid n$ respectively, at comparable sizes). More precisely, $|\Sigma_\chi(n, 3)|\sqrt{\log n}/n^2$ converges, for each residue of $n \pmod{3}$, to a constant expressible as a Bernays-type density integrated against the boundary geometry of the positivity simplex.

Evidence and proof strategy. On the interior range $N \leq Y$ the realized sets have counts $B(Y/3)$ versus $B(Y) - B(Y/3)$, whose ratio tends to $\frac{1}{2} (= \frac{1/3}{2/3})$ by (4); the boundary range $Y < N \leq (n-3)^2$, where only some directions of a representation fit in the simplex, contributes proportionally to both classes. Making this rigorous requires counting integers with *at least*

one representation in a prescribed sector, i.e. Hecke equidistribution of Eisenstein primes in angular sectors; this is classical technology but the uniformity needed here deserves care, so we record the statement as a conjecture. The measured ratio of ratios is 0.698, 0.683, 0.674 at $n \approx 300, 900, 2400$: drifting downward toward $\frac{1}{2}$ at the expected $1/\sqrt{\log n}$ crawl.

5 The spectrum and multiplicities for $k \geq 4$

5.1 The top of the spectrum: resolution of [4, Question 10.3]

Theorem 5.1. *Let $n \geq k + 2$.*

- (a) *For every $k \geq 3$, the two largest elements of $\Sigma_\chi(n, k)$ differ by exactly 1.*
- (b) *For $k = 2$ and $n \geq 4$, they differ by 1 if n is even and by 2 if n is odd.*

Proof. The largest edge count corresponds to the unique minimizer of s , the Turán partition T ([4, Theorem 4.4]), whose parts take the values $q = \lfloor n/k \rfloor$ and $q+1$. By Lemma 2.2 all spectrum s -values share the parity of n , so the second-smallest realizable s is at least $s(T) + 2$; it suffices to realize $s(T) + 2$.

(a) For $k \geq 3$ the parts of T take at most two values among $k \geq 3$ parts, so two parts are equal, say both of size v . If $v \geq 2$, transfer one unit between them: the new partition has positive parts and s increases by $(v+1)^2 + (v-1)^2 - 2v^2 = 2$. If $v = 1$ (which forces $q = 1$, i.e. $n < 2k$), then T has $n - k \geq 2$ parts of size 2 (as $n \geq k + 2$), and transferring one unit between two parts of size 2 gives parts 3, 1 and again $\Delta s = 2$.

(b) For $k = 2$: if $n = 2q$, the transfer $(q, q) \mapsto (q+1, q-1)$ gives $\Delta s = 2$, hence gap 1. If $n = 2q+1$, the partitions are $(a, n-a)$ and the two smallest s -values are at $a = q$ and $a = q-1$, with difference $(q-1)^2 + (q+2)^2 - q^2 - (q+1)^2 = 4$, hence gap 2 (matching [4, Theorem 6.4]). $\square$

5.2 Spectrum size: resolution of [4, Question 8.2]

Since $\Sigma_\chi(n, k) \subseteq [\text{sat}_\chi(n, k), \text{ab}_\chi(n, k)]$, an interval of length $\Delta_\chi(n, k) \leq \frac{k-1}{2k} n^2$ [4, Section 5], we always have the trivial bound

$$|\Sigma_\chi(n, k)| \leq \frac{k-1}{2k} n^2 + 1. \quad (5)$$

In particular the growth rate $\Theta(n^{k-1})$ contemplated in [4, Question 8.2] is impossible for every $k \geq 4$; the true rate is quadratic. The key to the lower bound is an explicit isometry.

Lemma 5.2 ($A_3 \cong D_3$). *The map $\varphi(v_1, v_2, v_3, v_4) = (v_1 + v_2, v_1 + v_3, v_1 + v_4)$ is a bijective isometry from $\Lambda_3 = \{v \in \mathbb{Z}^4 : \sum v_i = 0\}$ onto $D_3 = \{u \in \mathbb{Z}^3 : u_1 + u_2 + u_3 \text{ even}\}$, i.e. $|\varphi(v)|^2 = |v|^2$.*

Proof. Using $v_2 + v_3 + v_4 = -v_1$,

$$|\varphi(v)|^2 = 3v_1^2 + 2v_1(v_2 + v_3 + v_4) + v_2^2 + v_3^2 + v_4^2 = v_1^2 + v_2^2 + v_3^2 + v_4^2.$$

The coordinate sum of $\varphi(v)$ is $3v_1 + (v_2 + v_3 + v_4) = 2v_1$, even, so φ maps into D_3 ; and given $u \in D_3$ with $\sum u_i = 2m$, the vector $v = (m, u_1 - m, u_2 - m, u_3 - m)$ is the unique preimage, and it lies in Λ_3 . $\square$

By Gauss–Legendre, the positive integers that are sums of three squares are exactly those not of the form $4^a(8b+7)$; note that a sum of three squares is even iff the three summands have even coordinate sum. Hence:

Corollary 5.3. *The set of norms of Λ_3 , equivalently of D_3 , is $\{h \in 2\mathbb{Z}_{\geq 0} : h \neq 4^a(8b+7)\}$. Among even integers up to H the excluded values number $H/24 + O(\log H)$, so the norm set has $\frac{11}{24}H + O(\log H)$ elements up to H .*

Theorem 5.4. *For every fixed $k \geq 5$ and all n , and for $k = 4$ whenever $4 \mid n$,*

$$|\Sigma_\chi(n, k)| = \Theta_k(n^2).$$

Proof. The upper bound is (5). For the lower bound, first let $k = 4$ and $n = 4q$. For $t \in \Lambda_3$ with $\max_i |t_i| \leq q-1$ the tuple $(q+t_1, \dots, q+t_4)$ is a partition of n into 4 positive parts with $s = 4q^2 + |t|^2$ (the cross term vanishes as $\sum t_i = 0$). By Corollary 5.3, $|t|^2$ ranges over all even $h \leq (q-1)^2$ with $h \neq 4^a(8b+7)$, using that any norm- h vector of D_3 , pulled back by Lemma 5.2, has all coordinates of absolute value $\leq \sqrt{h} \leq q-1$. Distinct h give distinct s , so $|\Sigma_\chi(4q, 4)| \geq \frac{11}{24}(q-1)^2 + O(\log q) \gg n^2$.

Now let $k \geq 5$. Choose a padding part $p \in \{1, 2, 3, 4\}$ and, if $k \geq 6$, additional parts equal to 1, so that the remaining weight $n' = n - (k-5) - p$ satisfies $4 \mid n'$ (possible: p ranges over a complete residue system mod 4). Partitions of the form $(1, \dots, 1, p, b_1, b_2, b_3, b_4)$ with (b_i) from the $k = 4$ construction on n' have $s = (k-5)p^2 + s_4$, injective in s_4 , giving $|\Sigma_\chi(n, k)| \gg_k n^2$. $\square$

Proposition 5.5. *For $k = 4$ and every n , $|\Sigma_\chi(n, 4)| \gg n^2/\sqrt{\log n}$.*

Proof. Partitions $(1, b_1, b_2, b_3)$ with (b_i) a partition of $n-1$ into three parts have $s = 1 + s_3$, so $|\Sigma_\chi(n, 4)| \geq |\Sigma_\chi(n-1, 3)|$, and Theorem 4.10 applies. $\square$

Conjecture 5.6. $|\Sigma_\chi(n, 4)| = \Theta(n^2)$ for all n .

Evidence and strategy. The construction of Theorem 5.4 around an unbalanced base $(q, \dots, q+1, \dots)$ produces the values of a positive ternary quadratic polynomial (equivalently, norms on a coset of Λ_3), for which elementary substitutions only ever expose a binary section — a genuine rank barrier, since counts of values of binary forms lose the factor $\sqrt{\log n}$ (Bernays). Representation theorems for ternary lattice cosets in the style of Duke and Schulze–Pillot [7, 8] assert that all sufficiently large integers satisfying the local conditions, outside finitely many square classes, are represented; this yields the conjecture, and the data (e.g. $|\Sigma_\chi(n, 4)|/n^2 = 0.115, 0.141, 0.163, 0.172$ at $n = 20, 40, 60, 80$, increasing) supports it. We prefer to keep the elementary and the analytic inputs separated, and record the general- n statement as a conjecture.

5.3 Three-squares gaps for $k = 4$

Theorem 5.7. *Let $k = 4$. For every partition a of n into 4 parts, the discrepancy $D(a) = 4s(a) - n^2$ is never of the form $4^c(8b+7)$ with $b, c \geq 0$. Equivalently, since $D = 3n^2 - 4n - 8e$ for an edge count e , no edge count e with $3n^2 - 4n - 8e \in \{4^c(8b+7)\}$ belongs to $\Sigma_\chi(n, 4)$.*

Proof. By Theorem 2.3, $w = 4a - n\mathbf{1} \in \Lambda_3$ has $|w|^2 = 4D$. By Lemma 5.2, $4D = |\varphi(w)|^2$ is a sum of three squares, so $4D \neq 4^a(8b+7)$ by Gauss–Legendre. If $D = 4^c(8b+7)$ then $4D = 4^{c+1}(8b+7)$, a contradiction. The formula $D = 3n^2 - 4n - 8e$ follows from $e = \binom{n}{2} - \frac{1}{2}s$ and $D = 4s - n^2$. $\square$

Corollary 5.8. $\Sigma_\chi(n, 4)$ never has asymptotically full density inside its congruence class of discrepancies. For example, if $4 \mid n$ then $D \equiv 0 \pmod{8}$ for every partition (as $s \equiv n \equiv 0 \pmod{2}$), and within the class $D \equiv 0 \pmod{8}$ the excluded values $\{4^c(8b+7) : c \geq 2\}$ have relative density $\frac{1}{12}$.

Observation 5.9. Exhaustively for $n \leq 80$, no realized discrepancy is of the excluded form (as the theorem requires), and at $n = 60$ the *only* congruence-admissible values $D \leq n^2/8$ missing from the realized set are 112, 240, 368, 448 — precisely the numbers $4^c(8b+7)$ in range. This suggests that in the bulk the Gauss–Legendre exclusions are the only obstructions; the completeness statement is part of Conjecture 5.6 (via ternary coset representation [8]).

5.4 Complete intervals at the top for large k

In the opposite direction, once the rank of the perturbation lattice reaches 4 the spectrum becomes locally gap-free. Recall $\text{ab}_\chi(n, k) = e(T(n, k))$ [4, Theorem 4.4].

Theorem 5.10. Let $k \geq 8$ and $k \mid n$, and write $q = n/k$. Then

$$\{\text{ab}_\chi(n, k) - j : 0 \leq j \leq (q-1)^2\} \subseteq \Sigma_\chi(n, k) :$$

the top segment of the spectrum is a complete integer interval of length $(q-1)^2 + 1 = \Theta(n^2/k^2)$.

Proof. By Lagrange, write $j = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$ with $0 \leq \alpha, \beta, \gamma, \delta \leq \sqrt{j} \leq q-1$. The vector $t = (\alpha, -\alpha, \beta, -\beta, \gamma, -\gamma, \delta, -\delta, 0, \dots, 0) \in \Lambda_{k-1}$ (here $k \geq 8$ is used) satisfies $|t_i| \leq q-1$, so $(q+t_1, \dots, q+t_k)$ is a partition of n with $s = kq^2 + |t|^2 = s(T) + 2j$, i.e. edge count $\text{ab}_\chi(n, k) - j$. $\square$

Observation 5.11. For $k \in \{5, 6, 7\}$ with $k \mid n$ the same interval was verified complete for all tested cases ($n \leq 42$); this amounts to the statement that the root lattices A_4, A_5, A_6 represent every even integer in the relevant range, which should follow from a finite local computation for these quaternary-and-larger forms. For $k = 4$ the interval is provably *incomplete* (Theorem 5.7), and for $k = 3$ it is Loeschian-sparse (Theorem 4.10): the transition at rank 4 is exactly the classical transition in the representation theory of quadratic forms.

Remark 5.12. Theorem 5.10 confines all gaps of $\Sigma_\chi(n, k)$, for $k \geq 8$, $k \mid n$, to the lower $\text{ab}_\chi - \Theta(n^2/k^2)$ portion of the spectrum, consistent with [4, Conjecture 10.1], which locates the largest gap at the very bottom. The conjecture itself remains open: the sparse region near $\text{sat}_\chi(n, k)$, where partitions are highly unbalanced and the representing sphere meets only a thin corner of the simplex, is not controlled by our methods.

5.5 The phase transition in maximal multiplicity

Lemma 5.13 (Point-count bound). Let q be a positive definite integral quadratic form in $d \geq 2$ variables. Then $\#\{v \in \mathbb{Z}^d : q(v) = R\} \ll_{q,\varepsilon} R^{d/2-1+\varepsilon}$ for $d \geq 3$, and $\ll_{q,\varepsilon} R^\varepsilon$ for $d = 2$.

Proof. For $d = 2$ this is classical: after clearing denominators, the number of representations of R by a fixed positive binary form is bounded by an ideal count in the corresponding quadratic order, hence by a divisor-type function of R , which is $\ll R^\varepsilon$ (see e.g. [9, Ch. 12] for $x^2 + y^2$; the general case is identical).

For $d \geq 3$, write $v = (v', t)$ and $q(v) = q'(v') + tL(v') + ct^2$ with q' positive definite in $d-1$ variables and L linear. Positivity forces $|t| \ll \sqrt{R}$. For fixed t , completing the square

shows that $\#\{v' : q(v', t) = R\}$ equals the number of representations of a single target $\ll R$ by q' on a translated lattice; clearing the bounded denominator of the translation reduces this to representations of a target $\ll R$ by an integral positive $(d-1)$ -ary form depending only on q . By induction (base $d-1=2$ above), each such count is $\ll R^\varepsilon$ if $d=3$ and $\ll R^{(d-1)/2-1+\varepsilon}$ if $d \geq 4$. Summing over the $O(\sqrt{R})$ values of t yields $\ll R^{1/2+\varepsilon}$ for $d=3$ and $\ll R^{d/2-1+\varepsilon}$ for $d \geq 4$. $\square$

Theorem 5.14 (Phase transition). *For fixed k and $n \rightarrow \infty$:*

- (i) $M(n, 2) = 1$ ([4, Theorem 6.4]);
- (ii) $\log M(n, 3) = (2 \log 2 + o(1)) \log n / \log \log n$ (Theorem 4.8);
- (iii) $c n \leq M(n, 4) \ll_\varepsilon n^{1+\varepsilon}$ for some $c > 0$ and all large n ;
- (iv) for each $k \geq 5$, $c_k n^{k-3} \leq M(n, k) \ll_{k,\varepsilon} n^{k-3+\varepsilon}$.

Proof. Lower bounds in (iii)–(iv): by pigeonhole and (5),

$$M(n, k) \geq \frac{p_k(n)}{|\Sigma_\chi(n, k)|} \geq \frac{p_k(n)}{\frac{k-1}{2k} n^2 + 1} \gg_k n^{k-3},$$

using $p_k(n) \sim n^{k-1}/(k!(k-1)!)$ [1]. Note these bounds are completely effective. Upper bounds: by Theorem 2.3, $m_{n,k}(e)$ is at most the number of $w \in \Lambda_{k-1}$ with $|w|^2 = kD \ll_k n^2$; writing the form in $d = k-1$ free coordinates (eliminate $w_k = -\sum_{i < k} w_i$) and applying Lemma 5.13 with $R \ll n^2$ gives $M(n, k) \ll R^{(k-1)/2-1+\varepsilon} \ll n^{k-3+2\varepsilon}$ for $k \geq 4$. $\square$

Observation 5.15. Measured maxima match the theorem closely and exhibit the transition vividly: for $n = 30, 60, 90, 120$ we found $M(n, 3) = 3, 4, 6, 6$; $M(n, 4) = 5, 12, 18, 29$ (close to linear, $\approx 0.22n$); and $M(n, 5) = 8, 30, 67, 120$, with $M(n, 5)$ within rounding of $n^2/120$ at every sampled point. For $k \geq 6$ the average multiplicity $p_k(n)/|\Sigma_\chi(n, k)| \asymp n^{k-3}$ matches the maximum in order of magnitude: multiplicities *concentrate*. By contrast, for $k=3$ the average is $\asymp \sqrt{\log n}$ (Corollary 4.11) while the maximum is quasi-polynomial.

Remark 5.16 (Where hard analytic number theory would enter). Closing the ε -gap at $k=4$ — i.e. deciding whether $M(n, 4) \gg n^{1+\delta}$ for some $\delta > 0$, or $M(n, 4) = n^{1+o(1)}$ sharply — is a class-number problem: representation numbers of ternary forms are governed by $h(-d)$, whose extreme values are controlled only through Siegel’s ineffective theorem, and placing the representations inside the positivity simplex requires equidistribution of lattice points on spheres, i.e. Duke’s theorem [7, 8]. It is pleasing that the elementary pigeonhole already meets these deep inputs to within n^ε .

6 The depth filtration: clique counts, symmetric functions, and Prouhet–Tarry–Escott

Collisions as defined so far are agreements of the first two power sums $p_1 = \sum a_i$, $p_2 = \sum a_i^2$. This is the second level of a natural filtration with graph-theoretic meaning.

Proposition 6.1. *The number of K_t -subgraphs of $K_{a_1, \dots, a_k}$ equals $e_t(a_1, \dots, a_k)$, the elementary symmetric polynomial ($1 \leq t \leq k$).*

Proof. A clique meets each part in at most one vertex, so a K_t is a choice of t parts and one vertex from each. $\square$

Definition 6.2. For distinct partitions $a \neq b$ of n into k parts, the *depth* of the pair is the largest r with $p_j(a) = p_j(b)$ for all $j \leq r$. A collision pair has depth ≥ 2 .

Proposition 6.3. *Let $a \neq b$ be partitions of n into k parts.*

- (a) *The pair has depth $\geq r$ iff K_a and K_b have equal K_t -counts for all $t \leq r$.*
- (b) *The depth is at most $k - 1$; equivalently, two complete k -partite graphs on n vertices with equal K_t -counts for all $t \leq k$ are isomorphic.*
- (c) *Depth- r pairs exist (for some n) iff the Prouhet–Tarry–Escott system of size k and degree r , $\sum a_i^j = \sum b_i^j$ ($1 \leq j \leq r$), has a solution in distinct integer multisets.*

Proof. (a) By Newton’s identities over $\mathbb{Q}$, agreement of $p_1, \dots, p_r$ is equivalent to agreement of $e_1, \dots, e_r$; apply Proposition 6.1. (b) $p_1, \dots, p_k$ determine $e_1, \dots, e_k$, hence the multiset. (c) The PTE property for degrees $\leq r$ is invariant under the translation $a_i \mapsto a_i + c$ (expand binomially), so any integer solution can be translated to have positive parts, becoming a pair of partitions of the common sum n . $\square$

Example 6.4. The smallest depth-3 pair for $k = 4$, found by exhaustive search, is

$$(1, 4, 5, 8) \quad \text{and} \quad (2, 2, 7, 7) \quad (n = 18),$$

with common $p_1 = 18$, $p_2 = 106$, $p_3 = 702$. Thus $K_{1,4,5,8}$ and $K_{2,2,7,7}$ are non-isomorphic graphs on 18 vertices with 109 edges and 252 triangles each, first distinguished by their K_4 -counts, $e_4 = 160$ versus 196. The classical ideal PTE pair $\{1, 5, 8, 12\}, \{2, 3, 10, 11\}$ gives another such pair at $n = 26$ (221 edges, 676 triangles, K_4 -counts 480 and 660).

Corollary 6.5. *For every $k \leq 10$ and for $k = 12$ there exist n and non-isomorphic complete k -partite graphs on n vertices whose K_t -counts agree for every $t \leq k - 1$.*

Proof. Ideal PTE solutions (degree $k - 1$, size k) are known for these sizes [3]; apply Proposition 6.3(c) with $r = k - 1$. $\square$

Remark 6.6. Two caveats keep this honest. First, the graphs in Example 6.4 are distinguished by their degree sequences (a vertex in part i has degree $n - a_i$, so the partition is recoverable from degrees); the content of the filtration is about clique profiles, not about hard instances of isomorphism. Second, part (c) shows our depth hierarchy is exactly the PTE problem in disguise; the novelty is the graph-theoretic interpretation and the *counting* theory that Sections 4–5 develop at depth 2. Whether ideal solutions exist for $k = 11$, hence whether Corollary 6.5 extends, is a famous open problem.

Question 6.7. *Develop the depth-3 analogue of Sections 4–5: for fixed $k \geq 4$, estimate the number of depth- ≥ 3 fibers on n , the maximal depth-3 multiplicity, and their densities. Heuristically each additional level of depth imposes one further equation, suggesting depth- ≥ 3 pair counts of order $n^{k-4+o(1)}$; even the $k = 4$ case (where the $n = 18$ example shows the count is nonzero for infinitely many n , by translation) seems untouched.*

7 Collision dynamics: moves on fibers

A fiber $\{a : s(a) = s\}$ is a clique under the relation “collides with,” so the interesting dynamical object is not that graph but a *move graph*: fix a set of natural operations preserving (n, s) and ask whether they connect each fiber. The candidates furnished by Section 3 are the local reflections ι_S (Definition 3.4), which include the global ι when $k \mid 2n$.

Observation 7.1. Exhaustively for all $n \leq 40$: the numbers of fibers of size ≥ 2 that are *not* connected by local reflection moves are

k	3	4	5	6
disconnected/nontrivial fibers	107/257	595/1351	877/2067	1025/2252

The first failures are: $k = 3$: $n = 16$, $\{(1, 6, 9), (3, 3, 10)\}$; $k = 4$: $n = 14$, $\{(1, 1, 5, 7), (2, 2, 2, 8)\}$. So reflections generate many collisions (Example 3.5) but provably not all.

For $k = 3$ the failure is completely explained by arithmetic. Identify $(x, y) \in \mathbb{Z}^2$ with $z = x - y\omega \in \mathbb{Z}[\omega]$, so that $Q(x, y) = z\bar{z}$.

Proposition 7.2. $\text{Aut}(\mathbb{Z}^2, Q) = \{z \mapsto uz\} \cup \{z \mapsto u\bar{z}\}$ over the six units u . Consequently two representations of N are Aut-equivalent iff the Eisenstein ideals $(z), (z')$ satisfy $(z') \in \{(z), (\bar{z})\}$; and for $k = 3$ every reflection move (necessarily the global ι , as any S with $|S| \geq 3$ is the whole index set) acts on difference vectors through $\text{Aut}(\mathbb{Z}^2, Q)$. Hence a fiber containing partitions whose difference vectors generate ideals that are neither equal nor conjugate is never reflection-connected.

Proof. The twelve listed maps are norm-preserving lattice automorphisms and the point group of the hexagonal lattice has order 12, giving the first claim; Aut-equivalence classes of norm- N vectors then correspond to ideals of norm N up to conjugation, since $\mathbb{Z}[\omega]$ is a PID and associates generate equal ideals. For $k = 3$ a local reflection must use $|S| = 3$, i.e. S is everything and the move is ι (requiring $3 \mid n$), which acts as $-1 \in \text{Aut}$ on difference vectors by Theorem 2.3. $\square$

Example 7.3. The first failure $n = 16$, $\{(1, 6, 9), (3, 3, 10)\}$ has $N = 49$: the difference vectors generate the ideals (π^2) and $(7) = (\pi\bar{\pi})$ for $7 = \pi\bar{\pi}$, which are neither equal nor conjugate. (Also $3 \nmid 16$, so no reflection move exists at all.) These *arithmetic collisions* dominate asymptotically: by Theorem 4.8 multiplicities grow like divisor functions, while symmetry supplies only bounded orbits.

The cure for $k = 3$ is to adjoin an arithmetic move. Say two ideals of norm N are related by a *conjugation swap* if one is obtained from the other by replacing one factor π_i by $\bar{\pi}_i$ in its prime factorization.

Proposition 7.4. Fix $3 \nmid n$ and an interior fiber in the sense of Theorem 4.5(b). The map sending a partition to the unordered pair $\{(z), (\bar{z})\}$ of ideals generated by its difference vector is a bijection from the fiber onto the set of norm- N ideals modulo conjugation, and conjugation swaps act transitively on that set. Hence reflections together with conjugation swaps connect every interior fiber for $k = 3$, $3 \nmid n$.

Proof. By the proof of Theorem 4.5, the fiber is in bijection with the W -orbits of representations in one congruence class, and the involution $v \mapsto -v$ pairs the two W -orbits inside each Aut-orbit while swapping the classes; so fiber elements correspond to Aut-orbits of representations,

i.e. (Proposition 7.2) to ideals modulo conjugation. Consistency check: the number of ideals of norm N is $E(N)$, the self-conjugate ones are those of the form (m) or $(m\lambda)$, of norms m^2 and $3m^2$; since $N \equiv 1 \pmod{3}$ only $N = m^2$ occurs, so the number of conjugation classes is $\frac{1}{2}(E(N) + [N = \square]) = \frac{1}{2}w(N)$, matching Theorem 4.5. Transitivity: write $N = 3^c \prod p_i^{v_i} \prod q_j^{2w_j}$ with $p_i \equiv 1, q_j \equiv 2 \pmod{3}$; an ideal of norm N is $\lambda^c \prod \pi_i^{\alpha_i} \bar{\pi}_i^{v_i - \alpha_i} \prod q_j^{w_j}$ with $0 \leq \alpha_i \leq v_i$, and single-coordinate steps $\alpha_i \mapsto \alpha_i \pm 1$ (conjugation swaps) connect the box $\prod [0, v_i]$. $\square$

Question 7.5. *Is there a natural complete move system for $k \geq 4$? Local reflections realize the automorphisms of A_{k-1} and its relevant sublattices, but Observation 7.1 shows most fibers need more. For $k = 4$ the missing moves should correspond to the class-group and spinor-genus structure of ternary forms; a partition-level operation realizing them would be a discrete analogue of the equidistribution inputs of [7, 8], and even a conjectural description would be interesting.*

8 Computational verification

All computations are exhaustive enumerations over partitions, implemented in short, independent Python programs (≈ 100 lines each); no randomness, sampling, or floating-point arithmetic is involved except in the reported ratios. We record what was checked, since several theorems were actively stress-tested against these searches.

- (1) *Multiplicity formula.* For $n \in \{9, 10, 11, 12, 13, 22, 30, 31, 32, 45, 46, 60, 61, 90, 91\}$, every fiber with $N \leq (n-3)^2/12$ (a range strictly larger than the proved interior bound) was compared against Theorem 4.5(b): 386 fibers, 0 mismatches.
- (2) *Congruence dichotomy.* For sampled n , all realized invariants satisfied $N \equiv n^2 \pmod{3}$ (Lemma 4.1).
- (3) *Predicted first collisions.* The first fibers of multiplicity 3 and 4 occur at $(n, N) = (27, 147)$ and $(36, 273)$, as predicted in Example 4.7.
- (4) *Top gap.* Theorem 5.1 was verified for all $2 \leq k \leq 7$ and $k+2 \leq n \leq 60$.
- (5) *Three-squares exclusion.* For all $n \leq 80$, no partition of n into 4 parts has discrepancy of the form $4^c(8b+7)$ (Theorem 5.7); at $n = 60$ the excluded numbers are the only admissible bulk values missing (Observation 5.9).
- (6) *Top intervals.* Theorem 5.10 was checked at $(k, n) \in \{(8, 24), (8, 32), (9, 27), (9, 36), (10, 40)\}$, and the analogous completeness for $k = 5, 6, 7$ at $(5, 30), (5, 40), (6, 36), (7, 42)$ (Observation 5.11).
- (7) *Density mirage.* The table of Observation 4.12 (n up to 2401).
- (8) *Connectivity.* The move-graph statistics of Observation 7.1 are exhaustive over all fibers with $n \leq 40, 3 \leq k \leq 6$; an earlier working conjecture of ours, that local reflections connect all fibers for $k \geq 4$, was *refuted* by this search (first counterexample $n = 14$) and replaced by the symmetry/arithmetic dichotomy of Section 7.
- (9) *Depth.* Example 6.4 is the outcome of an exhaustive search over $n \leq 26$, and all power sums and clique counts quoted there were recomputed directly.

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